

SELF-DUAL AND LOGARITHMIC REPRESENTATIONS OF THE TWISTED HEISENBERG–VIRASORO ALGEBRA AT LEVEL ZERO

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ABSTRACT. This paper is a continuation of [2]. We present certain new applications and generalizations of the free field realization of the twisted Heisenberg–Virasoro algebra \mathcal{H} at level zero. We find explicit formulas for singular vectors in certain Verma modules. A free field realization of self-dual modules for \mathcal{H} is presented by combining a bosonic construction of Whittaker modules from [7] with a construction of logarithmic modules for vertex algebras. As an application, we prove that there exists a non-split self-extension of irreducible self-dual module which is a logarithmic module of rank two. We construct a large family of logarithmic modules containing different types of highest weight modules as subquotients. We believe that these logarithmic modules are related with projective covers of irreducible modules in a suitable category of \mathcal{H} -modules.

1. INTRODUCTION

The twisted Heisenberg–Virasoro Lie algebra \mathcal{H} is an important example of a Lie algebra whose associated vertex algebra has many applications in the representation theory and conformal field theory. If the level of the corresponding Heisenberg vertex subalgebra is non-zero, the Heisenberg–Virasoro vertex algebra is isomorphic to the tensor product of Heisenberg vertex algebra and the (universal or simple) Virasoro vertex algebra (cf. [1],[14]). The study of the twisted Heisenberg–Virasoro algebra at level zero was initiated by Y. Billig [12] motivated by applications to the toroidal Lie algebras. New results on the representation theory were obtained in recent papers [2], [18], [22].

In this paper we continue our study of free field realization of the twisted Heisenberg–Virasoro algebra from [2]. Our main motivation is to present free field realization of self-dual modules and certain Verma modules which we were unable to construct using methods from [2].

Let \mathcal{H} be the twisted Heisenberg–Virasoro algebra at level zero. Let $V^{\mathcal{H}}(h, h_I)$ (resp. $L^{\mathcal{H}}(h, h_I)$) denote the Verma module (resp. the irreducible highest weight module) with highest weight $(c_L, c_I, c_{L,I}, h, h_I)$ and highest weight vector v_{h, h_I} . Then $V^{\mathcal{H}}(h, h_I)$ is reducible if and only if $h_I/c_{L,I} - 1 \in \mathbb{Z} \setminus \{0\}$ (cf. [12]).

A free field realization of irreducible highest weight modules for the twisted Heisenberg–Virasoro algebra at level zero was presented by the authors in [2]. Using free field realization we calculated the fusion rules for non-generic irreducible modules i.e., for those irreducible modules which are not isomorphic to a Verma \mathcal{H} -module. In our approach, the screening operator Q introduced in [2, Section 2] has played an important role. In particular, the

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singular vector in $V^{\mathcal{H}}(h, h_I)$ in the case

$$h_I/c_{L,I} - 1 = p \in \mathbb{Z}_{>0}$$

is expressed as $Qv_{h+p, h_I} = S_p(c)v_{h, h_I}$ where $S_p(c)$ is a Schur polynomial in $c = -I/c_{L,I}$ (cf. [2, Theorem 4.3]).

In the present paper we found a similar approach for the singular vector in the case

$$(1.1) \quad h_I/c_{L,I} - 1 = -p \quad (p \in \mathbb{Z}_{>0}).$$

The formula is

$$(1.2) \quad \sum_{i=1}^p (L(-i)S_{p-i}(-c))v_{h, h_I} + \left(h + \frac{c_L - 2}{24}(p - 1)\right) S_p(-c)v_{h, h_I} - \frac{c_L - 26}{24} \left(\sum_{i=1}^p (i - 1)c(-i)S_{p-i}(-c)\right)v_{h, h_I}.$$

Verma modules $V^{\mathcal{H}}(h, h_I)$ in the case (1.1) are explicitly constructed in Section 5 by using certain deformation studied previously in [4]. Singular vector (1.2) admits a nice interpretation as an element $e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c+c}$ of the group algebra in the explicit lattice realization (cf. Theorem 5.3).

It turned out that the methods from [2] do not provide a free field realization of self-dual modules $L^{\mathcal{H}}(h, h_I)$ such that $h \neq \frac{c_L - 2}{24}$, $h_I = c_{L,I}$. On the other hand, the same paper contains certain results on vanishing of the fusion rules in the category of modules which contains self-dual module $L^{\mathcal{H}}(h, h_I)$ as above (see [2, Theorem 5.4(3), Remark 6.5]). In order to understand these fusion rules properties from the vertex-algebraic point of view, one needs to find a bosonic realization of self-dual modules. We shall see that the realization includes both the concepts of Whittaker and logarithmic modules for vertex algebras. We shall prove that the Whittaker module Π_{λ} for the vertex algebra $\Pi(0)$ introduced in [7], after certain logarithmic deformation (cf. Theorem 5.2) becomes a highly reducible \mathcal{H} -module $\tilde{\Pi}_{\lambda}$ which contains $L^{\mathcal{H}}(h, h_I)$ as a submodule. Since $\tilde{\Pi}_{\lambda}$ is not a module for the Heisenberg vertex algebra $M(1)$, it is clear that such module could not have appeared in the fusion rules analysis made in [2].

In Section 2 we recall from [2] a free field realization of \mathcal{H} , the definition of vertex algebra $\Pi(0)$ and its modules $\Pi(p, r)$. We present an extension of the Heisenberg–Virasoro vertex algebra $\overline{\Pi(0)} \subset \Pi(0)$ and give a structure of $\overline{\Pi(0)}$ -modules $\Pi(p, r)$ in Section 3. By using a certain relation in $\Pi(0)$ -modules we recover formula 1.2 in Section 4. Then we consider a deformed action of \mathcal{H} on these modules and obtain a family of modules $\widetilde{\Pi(p, r)}$ [cf. Theorem 5.2, pg. 9; Theorem 5.5, pg. 12] with the following properties:

- $\widetilde{\Pi(p, r)}$ is a logarithmic \mathcal{H} -module with the following action of the element $\widetilde{L(0)}$ of the Virasoro algebra:
- $\mathbb{C}[\widetilde{L(0)}]v$ is finite-dimensional for every $v \in \widetilde{\Pi(p, r)}$, $p \geq 1$,
- $\mathbb{C}[\widetilde{L(0)}]v$ is infinite-dimensional for every $v \in \Pi(p, r)$, $p \leq 0$.

- For $p \geq 1$, $\widetilde{\Pi(p, r)}$ admits a filtration

$$\widetilde{\Pi(p, r)} \cong \bigcup_{m \geq 0} Z^{(m)},$$

such that $Z^{(m)}$ is a logarithmic \mathcal{H} -module of rank $m+1$ and $Z^{(m)}/Z^{(m-1)}$ is a weight \mathcal{H} -module. [cf. Theorem 5.4, pg. 11]

In Section 6 we consider a deformed action on Whittaker $\Pi(0)$ -modules and obtain a realization of self-dual \mathcal{H} -modules $L^{\mathcal{H}}(h, c_{L,I})$, $h \neq \frac{c_L-2}{24}$ [cf. Theorem 6.3, pg. 14].

Finally, we find new applications of our results on the vertex algebra associated to the $W(2, 2)$ -algebra. We present a non-local bosonic formula (7.25) for the screening operator introduced by the authors in [3]. We hope that our new expression for screening operator can be applied to construction of a quantum group which would play the role of Kazhdan–Lusztig dual of the vertex algebra $W(2, 2)$.

We also discuss a connection of our approach with a realization of the BMS_3 -algebra obtained in [11] in the case $c_L = 26$.

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2. RESULTS FROM [2]

In this section we recall from [2] the free field realization of the twisted Heisenberg–Virasoro algebra, the construction of vertex algebra $\Pi(0)$ and their modules $\Pi(p, r)$. The definition of the lattice is slightly changed, but the action of the generators of the Heisenberg–Virasoro algebra is the same as in [2]. Main results of Section 2 of [2] are stated in Proposition 2.1.

We also present new, explicit formulas for sub-singular vectors introduced in [2, Proposition 2.7].

Recall that the twisted Heisenberg–Virasoro algebra is an infinite-dimensional complex Lie algebra \mathcal{H} with basis

$$\{L(n), I(n) : n \in \mathbb{Z}\} \cup \{C_L, C_{LI}, C_I\}$$

and commutation relation:

$$(2.3) \quad [L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3-n}{12} C_L,$$

$$(2.4) \quad [L(n), I(m)] = -mI(n+m) - \delta_{n,-m}(n^2+n)C_{LI},$$

$$(2.5) \quad [I(n), I(m)] = n\delta_{n,-m}C_I,$$

$$(2.6) \quad [\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0.$$

Let $V^{\mathcal{H}}(c_L, c_I, c_{L,I}, h, h_I)$ denote the Verma module with highest weight $(c_L, c_I, c_{L,I}, h, h_I)$, and $L^{\mathcal{H}}(c_L, c_I, c_{L,I}, h, h_I)$ be its irreducible quotient (cf. [12]). In this paper we consider the case $c_I = 0$. For simplicity we shall denote the Verma module $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ with $V^{\mathcal{H}}(h, h_I)$ and its simple quotient with $L^{\mathcal{H}}(h, h_I)$.

Define the following hyperbolic lattice $L = \mathbb{Z}c + \mathbb{Z}d$ such that

$$\langle c, c \rangle = \langle d, d \rangle = 0, \quad \langle c, d \rangle = 2.$$

Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the form $\langle \cdot, \cdot \rangle$ on \mathfrak{h} . We can consider \mathfrak{h} as an abelian Lie algebra. Let $\widehat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}K$ be the affinization of \mathfrak{h} . Let $\gamma \in \mathfrak{h}$ and consider $\widehat{\mathfrak{h}}$ -module

$$M(1, \gamma) := U(\widehat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}K)} \mathbb{C}$$

where $t\mathbb{C}[t] \otimes \mathfrak{h}$ acts trivially on \mathbb{C} , \mathfrak{h} acts as $\langle \delta, \gamma \rangle$ for $\delta \in \mathfrak{h}$ and K acts as 1. We shall denote the highest weight vector in $M(1, \gamma)$ by e^γ .

We shall write $M(1)$ for $M(1, 0)$. For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$ we write $h(n)$ for $t^n \otimes h$. Set $h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1}$. Then $M(1)$ is a vertex algebra which is generated by the fields $h(z)$, $h \in \mathfrak{h}$. Moreover, $M(1, \gamma)$ for $\gamma \in \mathfrak{h}$, are irreducible $M(1)$ -modules.

Define the Heisenberg and the Virasoro vector:

$$(2.7) \quad I = -c_{L,I}c(-1),$$

$$(2.8) \quad \omega = \frac{1}{2}c(-1)d(-1) + \frac{c_L - 2}{24}c(-2) - \frac{1}{2}d(-2).$$

Then the components of the fields

$$I(z) = Y(I, z) = \sum_{n \in \mathbb{Z}} I(n)z^{-n-1}, \quad L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$$

satisfy the commutation relations for the twisted Heisenberg–Virasoro Lie algebra \mathcal{H} and I and ω generate the simple Heisenberg–Virasoro vertex algebra $L^{\mathcal{H}}(c_L, 0, c_{L,I}, 0, 0)$ which we shall denote by $L^{\mathcal{H}}(c_L, c_{L,I})$.

We have:

$$(2.9) \quad [L(n), c(m)] = -mc(n+m) + (m^2 - m)\delta_{m+n,0}$$

$$(2.10) \quad [L(n), d(m)] = -md(n+m) - \frac{c_L - 2}{12}(m^2 - m)\delta_{m+n,0}$$

Let $V_L = M(1) \otimes \mathbb{C}[L]$ be the vertex algebra associated to the lattice L , where $\mathbb{C}[L]$ is the group algebra of L . Let $u = e^c$. Then

$$Q = \text{Res}_z Y(u, z) = u_0.$$

is a screening operator. One can show that

$$\Pi(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c]$$

is a simple vertex algebra (cf. [13]). Let Y be the associated vertex operator. The vertex operator $Y(u, z)$ and its component Q are well-defined on every $\Pi(0)$ -module.

Let

$$d^1 = d + \frac{c_L - 26}{12}c, \quad d^2 = d - \frac{c_L - 26}{12}c.$$

Consider the following irreducible $\Pi(0)$ -modules

$$\Pi(p, r) := \Pi(0) \cdot e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \quad \text{where } p \in \mathbb{Z}, r \in \mathbb{C}.$$

(The irreducibility of $\Pi(p, r)$ was also proved in [13]).

Let

$$(2.11) \quad \begin{aligned} h_{p,r} &= (1 - p^2) \frac{c_L - 26}{24} + 1 - p + (1 - r) \frac{p}{2}. \\ W_{p,r} &= U(\mathcal{H}) \cdot e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \subset \Pi(p, r). \end{aligned}$$

Set $v_{p,r} = e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c}$. Note that $h_{p,r+2} = h_{p,r} - p$.

The following result is proved in Propositions 2.5 and 2.7 of [2].

Proposition 2.1. *We have:*

- (1) $W_{p,r} \cong V^{\mathcal{H}}(h_{p,r}, (1-p)c_{L,I})$ if and only if $p \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$.
- (2) $W_{p,r} \cong L^{\mathcal{H}}(h_{p,r}, (1-p)c_{L,I})$ if $p \in \mathbb{Z}_{\geq 1}$.
- (3) For every $r \in \mathbb{C}$, $W_{0,r} \cong L^{\mathcal{H}}(\frac{c_L-2}{24}, c_{L,I})$.

The following modules were not constructed in [2]:

- Reducible Verma modules $V^{\mathcal{H}}(h_{p,r}, (1-p)c_{L,I})$ in the case $p \in \mathbb{Z}_{\geq 1}$.
- Self-dual modules $L^{\mathcal{H}}(h, c_{L,I})$ for $h \neq \frac{c_L-2}{24}$.

We shall present the construction of these modules in Sections 5 and 6.

We shall also need the following result.

Lemma 2.2. *Assume that $p \in \mathbb{Z}_{>0}$. As an \mathcal{H} -module, $\Pi(p, r)$ is generated by a family of \mathcal{H} -singular vectors $\{v_{p,r-2\ell} \mid \ell \in \mathbb{Z}\}$ and a family of \mathcal{H} -cosingular vectors $\{w_{p,r-2\ell}^{(m)} \mid \ell, m \in \mathbb{Z}, m \geq 1\}$, where*

$$v_{p,r-2\ell}^{(m)} = \frac{1}{2^m m!} d(-p)^m e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c + \ell c}.$$

Proof. By [2, Proposition 2.7] we have that $\Pi(p, r)$ is generated as \mathcal{H} -module by a family of singular vectors $\{v_{p,r-2\ell} \mid \ell \in \mathbb{Z}\}$ and by a family of cosingular vectors $\{w_{p,r-2\ell}^{(m)} \mid \ell, m \in \mathbb{Z}, m \geq 1\}$ satisfying

$$(2.12) \quad Q^m w_{p,r-2\ell}^{(m)} = v_{p,r-2(\ell+m)}.$$

Let us prove that for cosingular vectors we may choose $w_{p,r-2\ell}^{(m)} = v_{p,r-2\ell}^{(m)}$.

Recall the commutator relations in $\Pi(0)$:

$$[d(-k), e_j^c] = 2e_{j-k}^c, \quad [e_j^c, c(-k)] = 0, \quad [e_0^c, e_j^c] = 0.$$

We get that $Qv_{p,r-2\ell}^{(m)}$ is a linear combination of $S_{(i-1)p}(c)v_{p,r-2(\ell+1)}^{(m-i)}$, $i = 1, \dots, m$. Therefore we have

$$(2.13) \quad Q^j v_{p,r-2\ell}^{(m)} = \begin{cases} v_{p,r-2(\ell+j)}^{(m-j)} \bmod \text{Ker}_{\Pi(p,r)} Q^m, & j < m, \\ v_{p,r-2(\ell+m)}^{(m)}, & j = m. \end{cases}$$

The proof follows. □

See Figure 1 on pg. 19 for structure of $\Pi(0)$ -module $\Pi(p, r)$.

Remark 2.3. *Note that $\Pi(p, r) = \Pi(p, s)$ if and only if $r - s \in 2\mathbb{Z}$. Moreover, $\Pi(1, 1) = \Pi(0)$ and $U(\mathcal{H})v_{1,1} \cong L^{\mathcal{H}}(c_L, c_{L,I})$.*

3. AN EXTENSION OF THE VERTEX ALGEBRA $L^{\mathcal{H}}(c_L, c_{L,I})$

Now we study modules $\Pi(p, r)$ in more details. There is a vertex subalgebra of $\Pi(0)$ which can be treated as an extension of the vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$:

$$\overline{\Pi(0)} = \text{Ker}_{\Pi(0)} Q.$$

As we shall see, modules $\Pi(p, r)$ carry a natural $\overline{\Pi(0)}$ -structure. We obtain a filtration by $\overline{\Pi(0)}$ -modules such that the subquotients are irreducible over $\overline{\Pi(0)}$.

Proposition 3.1. *Let $\Pi(0)^{(n)} = \text{Ker}_{\Pi(0)} Q^{n+1}$. Then we have*

(1) $\overline{\Pi(0)}$ is a vertex subalgebra of $\Pi(0)$ which is as $L^{\mathcal{H}}(c_L, c_{L,I})$ -module isomorphic to

$$\overline{\Pi(0)} \cong \bigoplus_{n \in \mathbb{Z}} W_{1,1-2n} \cong \bigoplus_{n \in \mathbb{Z}} L^{\mathcal{H}}(n, 0).$$

(2) For every $n \in \mathbb{Z}_{\geq 0}$, $\Pi(0)^{(n)}$ and $\Pi(0)^{(n+1)}/\Pi(0)^{(n)}$ are $\overline{\Pi(0)}$ -modules. Moreover we have

$$\Pi(0) = \bigcup_{n \geq 0} \Pi(0)^{(n)}, \quad \Pi(0)^{(n)} \cdot \Pi(0)^{(m)} \subset \Pi(0)^{(n+m)}.$$

Proof. Since Q is a screening operator, $\overline{\Pi(0)} = \text{Ker}_{\Pi(0)} Q$ is a vertex subalgebra of $\Pi(0)$. By using [2, Proposition 2.7] we get

$$\begin{aligned} \overline{\Pi(0)} &= \bigoplus_{n \in \mathbb{Z}} L^{\mathcal{H}}(c_L, c_{L,I}) \cdot e^{nc} \\ &= \bigoplus_{n \in \mathbb{Z}} W_{1,1-2n}. \end{aligned}$$

The proof of assertion (2) is clear. \square

Condition (2) from Proposition 3.1 shows that \mathcal{H} -modules (and $\overline{\Pi(0)}$ -modules) $\Pi(0)^{(n)}$ give a $\mathbb{Z}_{\geq 0}$ filtration on the vertex algebra $\Pi(0)$. In the same way we can construct a filtration on certain $\Pi(0)$ -modules.

Theorem 3.2. *Assume that $p \in \mathbb{Z}_{>0}$. Let $\Pi(p, r)^{(m)} = \text{Ker}_{\Pi(p, r)} Q^{m+1}$. Then we have*

- (1) $\Pi(p, r) \cong \bigcup_{m \geq 0} \Pi(p, r)^{(m)}$, $\Pi(0)^{(n)} \cdot \Pi(p, r)^{(m)} \subset \Pi(p, r)^{(n+m)}$.
- (2) For every $m \in \mathbb{Z}_{\geq 0}$ $\Pi(p, r)^{(m)}$ is $\overline{\Pi(0)}$ -module and $\Pi(p, r)^{(m)} \subset \Pi(p, r)^{(m+1)}$.
- (3) $\Pi(p, r)^{(m+1)}/\Pi(p, r)^{(m)}$ is an irreducible $\overline{\Pi(0)}$ -module which is as \mathcal{H} -module isomorphic to

$$(3.14) \quad \bigoplus_{n \in \mathbb{Z}} W_{p, r-2n} \cong \bigoplus_{n \in \mathbb{Z}} L^{\mathcal{H}}(h_{p, r} + np, (1-p)c_{L, I}).$$

Proof. The proof of assertions (1) and (2) is clear. The decomposition (3.14) essentially follows from [2, Proposition 2.7]. Let us prove the irreducibility result in (3). It suffices to prove that

$$W_{1,1-2n} \cdot W_{p, r-2\ell} = W_{p, 1-2(n+\ell)}.$$

Recall that

$$W_{1,1-2n} = U(\mathcal{H}) \cdot e^{nc}, \quad W_{p, r-2\ell} = U(\mathcal{H}) \cdot u_{p, r-2\ell} \mod \Pi(p, r)^{(m)},$$

where $u_{p, r-2\ell}$ is an \mathcal{H} -cosingular vector in $\Pi(p, r)$ such that $Q^{m+1}u_{p, r-2\ell} = v_{p, r-2\ell}$.

Since $e_{k_0}^{nc}v_{p, r-2\ell} = v_{p, r-2(\ell+n)}$ for $k_0 = -n(p-1) - 1$ we get that

$$Q^{m+1}e_{k_0}^{nc}u_{p, r-2\ell} = v_{p, r-2(\ell+n)} \neq 0.$$

So $e_{k_0}^{nc}u_{p, r-2\ell} + \Pi(p, r)^{(m)}$ generates $W_{p, 1-2(n+\ell)}$ inside of $\Pi(p, r)^{(m+1)}/\Pi(p, r)^{(m)}$. The proof follows. \square

Figure 1 on pg. 19 represents a portion of module $\Pi(p, r)$ with action of Q and e_{-p}^c on (sub)singular generators obtained in Lemma 2.2. Quotient module $\Pi(p, r)^{(m+1)}/\Pi(p, r)^{(m)}$ is a direct sum of 'slices', each generated by $v_{p, r-2\ell}^{(m+1)} + \Pi(p, r)^{(m)}$ and isomorphic to $W_{p, r-2(\ell+m+1)}$.

Let us now consider modules $\Pi(-p, r)$. Recall that $h_{-p, r+2} = h_{-p, r} + p$.

Theorem 3.3.

(1) As an \mathcal{H} -module $\Pi(0, r)$ is isomorphic to

$$\bigoplus_{n \in \mathbb{Z}} W_{0, r} \cong \bigoplus_{n \in \mathbb{Z}} L^{\mathcal{H}} \left(\frac{c_L - 2}{24}, c_{L, I} \right).$$

(2) Let $p \in \mathbb{Z}_{>0}$. Consider Q as $\overline{\Pi(0)}$ -endomorphism of $\Pi(-p, r)$, and let $\Pi(-p, r)^{(m)} = \text{Im } Q^m$. Then for $m \in \mathbb{Z}_{>0}$ we have

- (a) $\Pi(-p, r) \cong \Pi(-p, r)^{(m)}$ as $\overline{\Pi(0)}$ -modules,
- (b) $\Pi(-p, r)^{(m)}/\Pi(-p, r)^{(m+1)}$ is an irreducible $\overline{\Pi(0)}$ -module which is as an \mathcal{H} -module isomorphic to

$$\bigoplus_{\ell \in \mathbb{Z}} L^{\mathcal{H}}(h_{-p, r} + \ell p, (1 + p)c_{L, I})$$

Proof. It was shown in [2] that $\Pi(p, r) \cong \bigoplus_{\ell \in \mathbb{Z}} W_{p, r+2\ell}$ as \mathcal{H} -module when $p \notin \mathbb{Z}_{>0}$. Decomposition in (1) then follows from Proposition 2.1 (3). Since

$$(3.15) \quad Qv_{-p, r} = S_p(c)v_{-p, r-2}$$

we see that $Q(W_{-p, r+2\ell}) \subset W_{-p, r+2(\ell-1)}$ and since Q commutes with the action of \mathcal{H} we have $\text{Ker } Q = 0$. Therefore $\Pi(-p, r) \cong \text{Im } Q = \Pi(-p, r)^{(1)}$, so claim (a) follows by iteration.

Let us prove assertion (b). It suffices to prove the claim for $m = 0$. General statement then follows from (a). Recall that $W_{-p, r+2\ell} \cong V^{\mathcal{H}}(h_{-p, r+2\ell}, (1 + p)c_{L, I})$ and notice that $Qv_{-p, r+2(\ell+1)} = S_p(c)v_{-p, r+2\ell}$ is an \mathcal{H} -singular vector in $W_{-p, r+2\ell}$ which generates the maximal submodule. Now $U(\mathcal{H}).(v_{-p, r+2\ell} + \text{Im } Q) \cong L^{\mathcal{H}}(h_{-p, r+2\ell}, (1 + p)c_{L, I})$. This proves the decomposition in (b). Proof of irreducibility in (b) is similar to the proof of Theorem 3.2. \square

See Figure 2 on pg. 19 for reference.

4. RELATIONS IN $\Pi(0)$ -MODULES

In this section we shall apply a relation in the vertex algebra $\Pi(0)$ on its modules and recover an explicit formula for a singular vector in the Verma module $V^{\mathcal{H}}(h, (1 - p)c_{L, I})$, for $p \geq 1$. We shall use this formula in Section 5 when we construct a free field realization of these Verma modules.

Let

$$e^{-c}(z) = Y(e^{-c}, z) = \sum_{i \in \mathbb{Z}} e_i^{-c} z^{-i-1}.$$

By direct calculation we get

$$(4.16) \quad L(-2)e^{-c} = \frac{c_L - 26}{24}c(-2)e^{-c} - \frac{1}{2}L(-1)(d(-1)e^{-c}).$$

Let

$$s = (L(-2) - \frac{c_L - 26}{24}c(-2))e^{-c}.$$

Then we get

Lemma 4.1. *On every $\Pi(0)$ -module we have*

$$\begin{aligned} \mathcal{Q} &= s_0 = \text{Res}_z Y(s, z) \\ &= \sum_{i=0}^{\infty} (L(-2-i)e_i^{-c} + e_{-i-1}^{-c}L(-1+i)) \\ &\quad - \frac{c_L - 26}{24} \sum_{i \in \mathbb{Z}} (i+1)c(-i-2)e_i^{-c} \\ &= 0. \end{aligned}$$

Proof. The assertion follows from (4.16) and the fact that

$$(L(-1)u)_0 = 0$$

in every vertex operator algebra. □

Now we shall see some consequences of the relation $\mathcal{Q} = 0$ for irreducible \mathcal{H} -modules $L^{\mathcal{H}}(h, h_I)$ such that $h_I = (1-p)c_{L,I}$, $p \in \mathbb{Z}_{>0}$, which are realized as $W_{p,r}$, for $r \in \mathbb{C}$.

We have

$$\begin{aligned} 0 &= \mathcal{Q} e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \\ &= \left(\sum_{i=0}^{\infty} (L(-2-i)e_i^{-c}) + e_{-1}^{-c}L(-1) + h_{p,r}e_{-2}^{-c} \right) e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \\ &\quad - \frac{c_L - 26}{24} \left(\sum_{i \in \mathbb{Z}} (i+1)c(-i-2)e_i^{-c} \right) e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \\ &= \left(\sum_{i=0}^{p-2} (L(-2-i)e_i^{-c}) + e_{-1}^{-c}L(-1) + h_{p,r}e_{-2}^{-c} \right) e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \\ &\quad - \frac{c_L - 26}{24} \left(\sum_{i=-2}^{p-2} (i+1)c(-i-2)e_i^{-c} \right) e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \\ &= \left(\sum_{i=1}^p L(-i)e_{i-2}^{-c} + \left(h_{p,r} - 1 + \frac{c_L - 26}{24}(p-1) \right) e_{-2}^{-c} \right. \\ &\quad \left. - \frac{c_L - 26}{24} \sum_{i=-1}^{p-2} (i+1)c(-i-2)e_i^{-c} \right) e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \\ (4.17) \quad &= \left(\sum_{i=1}^p L(-i)S_{p-i}(-c) + \left(h_{p,r} - 1 + \frac{c_L - 26}{24}(p-1) \right) S_p(-c) \right. \\ &\quad \left. - \frac{c_L - 26}{24} \sum_{i=1}^p (i-1)c(-i)S_{p-i} \right) e^{\frac{p-1}{2}d^2 + \frac{1-(r+2)}{2}c} \end{aligned}$$

In this way we have proved:

Proposition 4.2. *Let $p \in \mathbb{Z}_{>0}$. In $W_{p,r+2}$ we have:*

$$\begin{aligned} 0 = & \left(\left(\sum_{i=1}^p L(-i) S_{p-i}(-c) \right) + \left(h_{p,r} - 1 + \frac{c_L - 26}{24} (p-1) \right) S_p(-c) \right) e^{\frac{p-1}{2}d^2 + \frac{1-(r+2)}{2}c} \\ & - \frac{c_L - 26}{24} \left(\sum_{i=1}^p (i-1)c(-i) S_{p-i}(-c) \right) e^{\frac{p-1}{2}d^2 + \frac{1-(r+2)}{2}c} \end{aligned}$$

In particular, let $h_I = (1-p)c_{L,I}$, $h = h_{p,r+2}$. Then the singular vector of level p in $V^{\mathcal{H}}(h, h_I)$ is $\Phi_p(L, c)v_{h, h_I}$ where

$$\begin{aligned} \Phi_p(L, c) := & \sum_{i=1}^p (L(-i) S_{p-i}(-c)) + S_p(-c) \left(L(0) + \frac{c_L - 26}{24} (p-1) \right) \\ & - \frac{c_L - 26}{24} \left(\sum_{i=1}^p (i-1)c(-i) S_{p-i}(-c) \right) \end{aligned}$$

5. DEFORMED ACTION OF \mathcal{H} ON WEIGHT $\Pi(0)$ -MODULES AND REALIZATION OF VERMA MODULES

As we noticed in Section 2, the free field realization from [2] does not provide realization of Verma modules $V^{\mathcal{H}}(h, (1-p)c_{L,I})$ and their singular vectors in the case $p \geq 1$. In order to understand these Verma modules, we shall use certain deformation of free field realization from [2]. We shall use the construction from [4] to deform the action of the twisted Heisenberg–Virasoro algebra on $\Pi(0)$ -modules (see also [8], [16]). Let

$$\Delta(u, z) = z^{u_0} \exp \left(\sum_{n=1}^{\infty} \frac{u_n}{-n} (-z)^{-n} \right).$$

First we recall a definition of logarithmic modules. More information about structure theory of logarithmic modules can be found in literature on logarithmic vertex operator algebras (see [6], [15], [17], [20], [21] and reference therein).

Definition 5.1. (1) *A module (M, Y_M) for the conformal vertex algebra with conformal vector ω is a logarithmic module of rank $m \in \mathbb{Z}_{\geq 1}$ if*

$$(L(0) - L_{ss}(0))^m = 0, \quad (L(0) - L_{ss}(0))^{m-1} \neq 0,$$

where $L_{ss}(0)$ is the semisimple part of $L(0)$.

(2) *If for every $m \in \mathbb{Z}_{\geq 1}$ $(L(0) - L_{ss}(0))^m \neq 0$ on M , we say that (M, Y_M) is a logarithmic module of infinite rank.*

Theorem 5.2. *For every $\Pi(0)$ -module $(M, Y_M(\cdot, z))$,*

$$(\widetilde{M}, \widetilde{Y}_{\widetilde{M}}(\cdot, z)) := (M, Y_M(\Delta(u, z) \cdot, z))$$

is a $\overline{\Pi(0)} := \text{Ker}_{\Pi(0)} Q$ -module. The action of Heisenberg–Virasoro algebra is

$$\begin{aligned} \widetilde{I}(z) &= \sum_{n \in \mathbb{Z}} \widetilde{I}(n) z^{-n-1} = \widetilde{Y}_{\widetilde{M}}(I, z) = I(z) \\ \widetilde{L}(z) &= \sum_{n \in \mathbb{Z}} \widetilde{L}(n) z^{-n-2} = \widetilde{Y}_{\widetilde{M}}(\omega, z) = L(z) + z^{-1} Y_M(u, z). \end{aligned}$$

In particular,

$$(5.18) \quad \widetilde{I(n)} = I(n), \quad \widetilde{L(n)} = L(n) + u_n.$$

and $\widetilde{L(0)} - \widetilde{L_{ss}(0)} = u_0 = Q$.

Recall the definition of $\Pi(0)$ -modules

$$\Pi(p, r) := \Pi(0).e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \quad \text{where } p \in \mathbb{Z}, r \in \mathbb{C}.$$

Then $\widetilde{\Pi(p, r)}$ are logarithmic \mathcal{H} -modules which are uniquely determined by the action (5.18).

We shall also consider the cyclic submodules:

$$\widetilde{W_{p,r}} = U(\mathcal{H}).e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c} \subset \widetilde{\Pi(p, r)}.$$

5.1. Case $h_I = (1-p)c_{L,I}$. We saw that for the undeformed action of \mathcal{H} studied in [2], vector $v_{p,r} = e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c}$, for $p \geq 1$, generates the irreducible highest weight module $W_{p,r}$. But we shall see below that $\widetilde{W_{p,r}}$ is isomorphic to a Verma module.

Theorem 5.3. *Assume that $p \in \mathbb{Z}_{>0}$, $r \in \mathbb{C}$. We have*

- (1) $\widetilde{W_{p,r}} \cong V^{\mathcal{H}}(h_{p,r}, (1-p)c_{L,I})$.
- (2) *Singular vectors in $\widetilde{W_{p,r}} \cong V^{\mathcal{H}}(h_{p,r}, (1-p)c_{L,I})$ are*

$$\text{Sing} = \{v_{p,r-2n} \mid n \geq 0\}.$$

Proof. It is clear that $v_{p,r-2n} = e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c + nc}$ is a singular vector for any $n \geq 0$. We only need to prove that

$$(5.19) \quad v_{p,r-2n} \in \widetilde{W_{p,r}}.$$

Assume that (5.19) holds for $n \in \mathbb{Z}_{\geq 0}$. Since $e_k^c v_{p,r-2n} = 0$ for every $k \geq 1-p$ we have

$$\widetilde{L(k)} = L(k) \quad \text{on } \mathbb{C}[c(-1), c(-2), \dots].e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c + nc}.$$

But since $e_{-p}^c v_{p,r-2n} = v_{p,r-2(n+1)}$ we have

$$\widetilde{L(-p)}v_{p,r-2n} = L(-p)v_{p,r-2n} + v_{p,r-2(n+1)}.$$

By using the expression for singular vector in $V^{\mathcal{H}}(h_{p,r}, (1-p)c_{L,I})$ from Proposition 4.2 we get for $h = h_{p,r-2n}$

$$\begin{aligned} & \Phi_p(\widetilde{L}, c).v_{p,r-2n} \\ = & \sum_{i=1}^p \left(\widetilde{L}(-i)S_{p-i}(-c) \right) + S_p(-c) \left(h + \frac{c_L - 2}{24}(p-1) \right) \\ & - \frac{c_L - 26}{24} \left(\sum_{i=1}^p (i-1)c(-i)S_{p-i}(-c) \right) v_{p,r-2n} \\ = & \Phi_p(L, c).v_{p,r-2n} + v_{p,r-2(n+1)} \\ = & v_{p,r-2(n+1)}. \end{aligned}$$

(Above we used the fact that $v_{p,r-2n}$ generates the irreducible \mathcal{H} -module $L^{\mathcal{H}}(h_{p,r}, (1-p)c_{L,I})$ for the undeformed action, so $\Phi_p(L, c).v_{p,r-2n} = 0$).

Thus we get that $v_{p,r-2(n+1)} = e^{\frac{p-1}{2}d^2 + \frac{1-r}{2}c + (n+1)c}$ belongs to $\widetilde{W}_{p,r}$. The claim now follows by induction. \square

Finally, we obtain a deformed version of Theorem 3.2.

Theorem 5.4. *Let $Z^{(m)} = \text{Ker } \widetilde{\Pi(p,r)} Q^{m+1}$. Then we have*

- (1) $\widetilde{\Pi(p,r)} \cong \bigcup_{m \geq 0} Z^{(m)}$, $\Pi(0)^{(n)} \cdot Z^{(m)} \subset Z^{(n+m)}$.
 - (2) For every $m \in \mathbb{Z}_{\geq 0}$, $Z^{(m)}$ is a logarithmic $\overline{\Pi(0)}$ -module of rank $m+1$ with respect to $\widetilde{L(0)}$.
 - (3) $Z^{(m)}/Z^{(m-1)}$ is a weight $\overline{\Pi(0)}$ -module which is as \mathcal{H} -module isomorphic to
- $$(5.20) \quad \bigcup_{n \in \mathbb{Z}} \widetilde{W}_{p,r-2n}.$$

Proof. Assertion (1) is clear. Using relation (2.13) in the proof of Lemma 2.2 we see that $v_{p,r-2\ell}^{(m)} \in Z^{(m)} \setminus Z^{(m-1)}$. Since $\widetilde{L(0)} - \widetilde{L_{ss}(0)} = Q$, we have that $Z^{(m)}$ is a logarithmic module of $\widetilde{L(0)}$ -nilpotent rank $m+1$ so (2) holds. Assertion (3) results from following facts:

- (a) As an \mathcal{H} -module $\widetilde{\Pi(p,r)}$ is generated by set of vectors

$$\{v_{p,r-2\ell} \mid \ell \in \mathbb{Z}\} \bigcup \{v_{p,r-2\ell}^{(m)} \mid m, \ell \in \mathbb{Z}, m \geq 1\}.$$

- (b) $Z^{(m)}/Z^{(m-1)}$ is a weight \mathcal{H} -module (i.e., non-logarithmic) generated by vectors $\{v_{p,r-2j}^{(m)} + Z^{(m-1)} \mid \ell \in \mathbb{Z}\}$.
- (c) $v_{p,r-2\ell}^{(m)} + Z^{(m-1)}$ generates the Verma module $\widetilde{W}_{p,r-2\ell}$.

Since Q and e_{-j}^c commute and by using (2.12) we get

$$Q^{(m)} e_{-j}^c v_{p,r-2\ell}^{(m)} = e_{-j}^c v_{p,r-2(\ell+m)} = S_{j-p}(c) v_{p,r-2(\ell+m)}$$

so $e_{-j}^c v_{p,r-2\ell}^{(m)} \in Z^{(m-1)}$ for $j < p$. Therefore

$$(5.21) \quad \widetilde{L(-j)} v_{p,r-2\ell}^{(m)} = \begin{cases} L(-j) v_{p,r-2\ell}^{(m)} \bmod Z^{(m-1)}, & j < p, \\ L(-p) v_{p,r-2\ell}^{(m)} + v_{p,r-2(\ell+1)}^{(m)}, & j = p. \end{cases}$$

The proof of claims (a) and (b) easily follows from Theorem 3.2 and (5.21). Let us prove claim (c).

We have proved in (5.21) that $v_{p,r-2\ell}^{(m)} + Z^{(m-1)}$ is a highest weight vector with highest weight $(h_{p,r-2\ell}, (1-p)c_{L,I})$. Now, repeating the same arguments as in the proof of Theorem 5.3 we get

$$\begin{aligned} & \Phi_p(\widetilde{L}, c) \cdot v_{p,r-2\ell}^{(m)} \\ &= \Phi_p(L, c) \cdot v_{p,r-2\ell}^{(m)} + v_{p,r-2(\ell+1)}^{(m)} \bmod Z^{(m-1)} \\ &= v_{p,r-2(\ell+1)}^{(m)} \bmod Z^{(m-1)}. \end{aligned}$$

This implies that $v_{p,r-2\ell}^{(m)} + Z^{(m-1)}$ generates the Verma module $\widetilde{W}_{p,r-2\ell}$ which contains all Verma modules $W_{p,r-2(\ell+j)}$, $j \in \mathbb{Z}_{\geq 1}$. This completes the proof. (See also Figure 3 on pg. 19 where one can follow steps in the proof.) \square

5.2. **Case** $h_I = c_{L,I}$. Note that $\widetilde{\Pi(0, r)}$ is an \mathcal{H} -module on which $I(0)$ acts as multiplication by $c_{L,I}$.

In particular, $\widetilde{W_{0,r}}$ is a $\mathbb{Z}_{\geq 0}$ -graded logarithmic \mathcal{H} -module whose lowest component is

$$\widetilde{W_{0,r}}(0) := \text{span}_{\mathbb{C}}\{v_{0,r-2\ell} \mid \ell \in \mathbb{Z}_{\geq 0}\}.$$

Moreover, since

$$\widetilde{L(0)} - \frac{c_L - 2}{24} = Q, \quad Q^n v_{0,r} = v_{0,r-2n}$$

we conclude that $\widetilde{W_{0,r}}$ is a $\mathbb{Z}_{\geq 0}$ -graded logarithmic module of infinite rank. See Figure 4 on pg. 20.

5.3. **Case** $h_I = (1+p)c_{L,I}$. We saw that $\Pi(-p, r)$ contains a descending chain of submodules $\Pi(-p, r)^{(m)}$ isomorphic to $\Pi(-p, r)$ (Theorem 3.3).

Theorem 5.5. *Let $p \in \mathbb{Z}_{>0}$. Then $\widetilde{\Pi(-p, r)}$ is a logarithmic $\overline{\Pi(0)}$ -module such that*

$$\dim_{\mathbb{C}}[\widetilde{L(0)}]v = \infty \quad \text{for every } v \in \widetilde{\Pi(-p, r)}.$$

Quotient $\widetilde{\Pi(-p, r)}/\widetilde{\Pi(-p, r)}^{(1)}$ is a weight module such that

$$\widetilde{L(n)}v_{-p,r} = S_{p-n}(c)v_{-p,r-2} \quad (1 \leq n \leq p) \quad \text{and} \quad \widetilde{L(n)}v_{-p,r} = 0 \quad (n > p).$$

Proof. Let $S = \{v_{-p,r-2\ell} \mid \ell \in \mathbb{Z}\}$. Let $\langle S \rangle$ be the \mathcal{H} -submodule generated by the set S . We shall first prove that $\widetilde{\Pi(-p, r)} = \langle S \rangle$. Since, as a vector space $\widetilde{\Pi(-p, r)} \cong \Pi(-p, r) \cong \bigoplus_{\ell \in \mathbb{Z}} W_{-p,r-2\ell}$, it suffices to show that $W_{-p,r-2\ell} \subset \langle S \rangle$ for each ℓ .

Take an arbitrary basis vector

$$u = c(-p_1) \cdots c(-p_s) L(-q_1) \cdots L(-q_m) v_{-p,r-2\ell}$$

of $W_{-p,r-2\ell}$, where $\ell \in \mathbb{Z}$, $p_1, \dots, p_s, q_1, \dots, q_m \geq 1$. Then by applying formula for $\widetilde{L(n)}$ and relation $[e_m^c, L(n)] = m e_{m+n}^c$ we get

$$\begin{aligned} & c(-p_1) \cdots c(-p_s) \widetilde{L(-q_1)} \cdots \widetilde{L(-q_m)} v_{-p,r-2\ell} \\ &= c(-p_1) \cdots c(-p_s) L(-q_1) \cdots L(-q_m) v_{-p,r-2\ell} + w \end{aligned}$$

where w is a linear combination of vectors

$$c(-t_1) \cdots c(-t_{s'}) L(-u_1) \cdots L(-u_{m'}) v_{-p,r-2\ell'}$$

such that $\ell' \in \mathbb{Z}$, $m' < m$. The assertion now follows by induction on m .

Furthermore, we have

$$(\widetilde{L(0)} - L(0))^n v_{-p,r} = Q^n v_{-p,r} = (S_p(c))^n v_{-p,r-2n}.$$

Since Q commutes with action of \mathcal{H} , we proved the first claim.

Taking a quotient by $\widetilde{\Pi(-p, r)}^{(1)} = \text{Im } Q$ results in a weight module (i.e., non logarithmic module) on which $\widetilde{L(0)} \equiv L(0)$. \square

See Figure 5 on pg. 20.

Remark 5.6. *As far as we know, modules $\widetilde{\Pi(-p, r)}/\widetilde{\Pi(-p, r)}^{(1)}$, and their cyclic submodules generated by images of $v_{-p, r-2\ell}$ are weight \mathcal{H} -modules which haven't been analysed in the literature.*

6. REALIZATION OF SELF-DUAL MODULES VIA WHITTAKER $\Pi(0)$ -MODULES

In Section 5 we slightly refined the free field realization from [2], but these results still don't give a realization of all irreducible self-dual modules. In order to construct all self-dual modules we shall apply the deformation from Section 5 on the Whittaker $\Pi(0)$ -module Π_λ which was constructed in [7, Section 11] and used for a realization of Whittaker $A_1^{(1)}$ -modules at the critical level. As a by-product we shall see that self-dual modules for \mathcal{H} have non-trivial self-extensions which are logarithmic modules.

6.1. Whittaker $\Pi(0)$ -module Π_λ . We shall recall the construction of a Whittaker $\Pi(0)$ -module Π_λ from [7, Section 11]. Let $u = e^c$, $u^{-1} = e^{-c}$.

Theorem 6.1. [7] *Assume that $\lambda \neq 0$. There is a $\Pi(0)$ -module Π_λ generated by the cyclic vector w_λ such that $c(0) = -\text{Id}$ on Π_λ and*

$$u_0 w_\lambda = \lambda w_\lambda, \quad u_0^{-1} w_\lambda = \frac{1}{\lambda} w_\lambda, \quad u_n w_\lambda = u_n^{-1} w_\lambda = 0 (n \geq 1).$$

As a vector space

$$\Pi_\lambda \cong \mathbb{C}[d(-n), c(-n-1) \mid n \geq 0] = \mathbb{C}[d(0)] \otimes M(1).$$

Π_λ is $\mathbb{Z}_{\geq 0}$ -graded

$$\Pi_\lambda = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Pi_\lambda(n)$$

and lowest component is isomorphic to $\mathbb{C}[d(0)]$.

Recall also (cf. [7]) that the lowest component $\Pi_\lambda(0)$ is an irreducible Whittaker module for the associative algebra \mathcal{A} defined by generators

$$d(0), e^{nc} \quad (n \in \mathbb{Z})$$

and relations

$$[d(0), e^{nc}] = 2ne^{nc}, \quad e^{nc}e^{mc} = e^{(n+m)c} \quad (n, m \in \mathbb{Z}).$$

6.2. Realization of self-dual modules. Now we can apply Theorem 5.2 on the Whittaker $\Pi(0)$ -module Π_λ . We get \mathcal{H} -module $\tilde{\Pi}_\lambda$, which is as a vector space isomorphic to Π_λ and the (deformed) action of \mathcal{H} is as follows:

$$(6.22) \quad \widetilde{I(0)} \equiv -c_{L,I}c(0) \equiv c_{L,I}\text{Id} \quad \text{on } \tilde{\Pi}_\lambda,$$

and on the lowest component $\tilde{\Pi}_\lambda(0)$ we have

$$(6.23) \quad \begin{aligned} \widetilde{L(0)} &\equiv \frac{1}{2}d(0)(c(0) + 1) - \frac{c_L - 2}{24}c(0) + u_0 \quad \text{on } \tilde{\Pi}_\lambda(0) \\ &\equiv \frac{c_L - 2}{24}\text{Id} + u_0 \quad \text{on } \Pi_\lambda(0). \end{aligned}$$

This implies:

$$(6.24) \quad \widetilde{I(0)}w_\lambda = c_{L,I}w_\lambda, \quad \widetilde{L(0)}w_\lambda = \left(\frac{c_L - 2}{24} + \lambda\right)w_\lambda.$$

Define also the following (logarithmic) cyclic module:

$$\widetilde{\Pi}_\lambda^{(n)} = U(\mathcal{H})d(0)^nw_\lambda.$$

Lemma 6.2. *We have:*

$$\widetilde{\Pi}_\lambda^{(n+1)} \supset \widetilde{\Pi}_\lambda^{(n)}, \quad \widetilde{\Pi}_\lambda = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \widetilde{\Pi}_\lambda^{(n)}.$$

Proof. By using (6.23) one can easily see that for $0 \leq m \leq n$ there is a polynomial $P(x)$ such that

$$P(\widetilde{L(0)})d(0)^nw_\lambda = d(0)^mw_\lambda.$$

This proves that $\widetilde{\Pi}_\lambda^{(n+1)} \supset \widetilde{\Pi}_\lambda^{(n)}$ for $n \in \mathbb{Z}_{\geq 0}$.

Take an arbitrary basis vector

$$u = c(-p_1) \cdots c(-p_s)d(-q_1) \cdots d(-q_r)d(0)^\ell w_\lambda$$

of $\widetilde{\Pi}_\lambda$, where $\ell \in \mathbb{Z}_{\geq 0}$, $p_1, \dots, p_s, q_1, \dots, q_r \geq 1$. Then

$$\begin{aligned} & c(-p_1) \cdots c(-p_s) \widetilde{L(-q_1)} \cdots \widetilde{L(-q_r)} d(0)^\ell w_\lambda \\ &= A c(-p_1) \cdots c(-p_s) d(-q_1) \cdots d(-q_r) d(0)^\ell w_\lambda + w \end{aligned}$$

where $A \neq 0$ and w is a linear combination of vectors

$$c(-t_1) \cdots c(-t_{s'}) d(-u_1) \cdots d(-u_{r'}) d(0)^{\ell'} w_\lambda$$

such that $\ell' \in \mathbb{Z}_{\geq 0}$, $r' < r$ or $r = r'$ and $u_1 + \cdots + u_{r'} < q_1 + \cdots + q_r$. The assertion now follows by induction. \square

Theorem 6.3. *For every $\lambda \in \mathbb{C}$, $\lambda \neq 0$ we have:*

- (1) $\widetilde{\Pi}_\lambda$ is a logarithmic \mathcal{H} -module of infinite rank with respect to $\widetilde{L(0)}$.
- (2) $\widetilde{\Pi}_\lambda^{(0)}$ is an irreducible self-dual \mathcal{H} -module with highest weight

$$(h, h_I) = \left(\frac{c_L - 2}{24} + \lambda, c_{L,I}\right).$$

- (3) \mathcal{H} -module $\widetilde{\Pi}_\lambda$ admits the $\mathbb{Z}_{\geq 0}$ -filtration:

$$\widetilde{\Pi}_\lambda = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \widetilde{\Pi}_\lambda^{(n)}$$

such that

$$\widetilde{\Pi}_\lambda^{(0)} = L^{\mathcal{H}}(h, h_I), \quad \widetilde{\Pi}_\lambda^{(n+1)} / \widetilde{\Pi}_\lambda^{(n)} \cong L^{\mathcal{H}}(h, h_I).$$

Every $\widetilde{\Pi}_\lambda^{(n)}$ is a logarithmic \mathcal{H} -module of rank n with respect to $\widetilde{L(0)}$.

Proof. (1) follows from the fact that on the top component $\widetilde{\Pi}_\lambda(0)$ we have $Q = u_0 = \widetilde{L(0)} - \frac{c_L - 2}{24}$.

(2) Using (6.24) and the fact that the Verma module with highest weight $(h, h_I) = (\frac{c_L - 2}{24} + \lambda, c_{L,I})$ is irreducible we get $L^{\mathcal{H}}(h, h_I) = U(\mathcal{H})w_\lambda = \widetilde{\Pi}_\lambda^{(0)}$.

(3) First we notice that for $m \geq 0$ we have

$$\widetilde{L(m)d(0)^{n+1}w_\lambda} = h\delta_{m,0}d(0)^{n+1}w_\lambda \mod \widetilde{\Pi}_\lambda^{(n)}.$$

Therefore we have isomorphism $L^\mathcal{H}(h, h_I) \rightarrow \widetilde{\Pi}_\lambda^{(n+1)}/\widetilde{\Pi}_\lambda^{(n)}$. The proof now follows from Lemma 6.2. \square

See Figure 6 on pg. 20 for reference.

We list two interesting consequences of previous theorem.

Corollary 6.4. *Logarithmic \mathcal{H} -module $\widetilde{\Pi}_\lambda^{(1)}$ is a non-split self-extension of irreducible self-dual module $L^\mathcal{H}(h, h_I)$:*

$$0 \rightarrow L^\mathcal{H}(h, h_I) \rightarrow \widetilde{\Pi}_\lambda^{(1)} \rightarrow L^\mathcal{H}(h, h_I) \rightarrow 0.$$

Note that the vertex algebra $\overline{\Pi(0)}$ is not $\mathbb{Z}_{\geq 0}$ -graded since for every $n \in \mathbb{Z}$, e^{nc} has weight n . Irreducible $\overline{\Pi(0)}$ -modules from Theorem 5.4 (3) are not $\mathbb{Z}_{\geq 0}$ -graded. But, quite surprisingly, the vertex algebra $\overline{\Pi(0)}$ admits a large family of $\mathbb{Z}_{\geq 0}$ -graded modules which are self-dual. We also construct a family of intertwining operators which haven't appeared in [2].

Corollary 6.5. *We have:*

- (1) $L^\mathcal{H}(h, c_{L,I})$ is an irreducible $\mathbb{Z}_{\geq 0}$ -graded $\overline{\Pi(0)}$ -module.
- (2) For every $n \in \mathbb{Z}$ there is a non-trivial intertwining operator of type

$$\begin{pmatrix} L^\mathcal{H}(h, c_{L,I}) \\ L^\mathcal{H}(n, 0) & L^\mathcal{H}(h, c_{L,I}) \end{pmatrix}.$$

7. SOME APPLICATIONS TO THE $W(2, 2)$ -ALGEBRA

In [2] we introduced a free field realization of the $W(2, 2)$ -algebra as a subalgebra of the Heisenberg Virasoro algebra.

Recall that $W(2, 2)$ is realized as a subalgebra of $L^\mathcal{H}(c_L, c_{L,I})$ generated by $L(z)$ and

$$W(z) = c_{L,I}^2 \overline{W}(z)$$

where

$$\overline{W}(z) = \sum_{n \in \mathbb{Z}} \overline{W}(n) z^{-n-2} = Y(c(-1)^2 - 2c(-2), z) = c(z)^2 - 2\partial c(z).$$

In the paper [3] we discussed a free field realization of highest weight $W(2, 2)$ -modules. We constructed in [3, Section 4] $W(2, 2)$ -homomorphism $S_1 : L^\mathcal{H}(c_L, c_{L,I}) \rightarrow L^\mathcal{H}(1, 0)$ such that $\text{Ker}_{L^\mathcal{H}(c_L, c_{L,I})} S_1$ is the simple vertex algebra $L^{W(2,2)}(c_L, -24c_{L,I}^2)$. In this paper shall present a bosonic, non-local expression for the screening operator S_1 .

The vertex algebra $W(2, 2)$ has appeared in physics literature as the Galilean Virasoro algebra ([23], [9], [10]) and as BMS₃ algebra ([11]). We noticed a free field realization of the $W(2, 2)$ -algebra in terms of the $\beta\gamma$ systems in [11]. We shall see how this realization relates to our approach.

7.1. A bosonic formula for the second screening operator and $W(2,2)$ -algebra.

Our approach is motivated by the realization of screening operators in LCFT from [5] and [6]. Recall the definition of modules $W_{p,r}$ from (2.11). For $r \in \mathbb{Z}$ we define:

$$\begin{aligned}
 S &= -\text{Res}_z \text{Res}_{z_1} \left(\text{Log}(1 - \frac{z_1}{z}) e^c(z) d^1(z_1) - \text{Log}(1 - \frac{z}{z_1}) d^1(z_1) e^c(z) \right) \\
 (7.25) \quad &= \sum_{j=1}^{\infty} \frac{1}{j} (d^1(-j) e_j^c - e_{-j}^c d^1(j)) : W_{p,r} \rightarrow W_{p,r-2}.
 \end{aligned}$$

Lemma 7.1. *We have:*

$$\begin{aligned}
 [L(m), S] &= d^1(m) e_0^c - e_m^c d^1(0) + 2\delta_{m,0} e_0^c, \\
 [c(m), S] &= 2e_m^c - 2\delta_{m,0} e_0^c, [W(m), S] = 0.
 \end{aligned}$$

Proof. In the proof we use the following formulas

$$(7.26) \quad [L(n), e_m^c] = -m e_{n+m}^c,$$

$$(7.27) \quad [L(n), d^1(m)] = -m d^1(n+m) - 2(m^2 - m) \delta_{m+n,0}.$$

We have:

$$\begin{aligned}
 [L(n), S] &= \sum_{j=1}^{\infty} \frac{-j}{j} (d^1(-j) e_{j+n}^c + e_{-j+n}^c d^1(j)) \\
 &\quad + \sum_{j=1}^{\infty} \frac{j}{j} (d^1(-j+n) e_{j+m}^c + e_{-j}^c d^1(j+n)) \\
 &\quad - 2 \sum_{j=1}^{\infty} \frac{j^2+j}{j} \delta_{-j+n,0} e_{j+m}^c + 2 \sum_{j=1}^{\infty} \frac{j^2-j}{j} \delta_{j+n,0} e_{-j}^c \\
 &= - \sum_{j=1}^{\infty} (d^1(-j) e_{j+n}^c + e_{-j+n}^c d^1(j)) + \sum_{j=1}^{\infty} (d^1(-j+n) e_j^c + e_{-j}^c d^1(j+n)) \\
 &\quad - 2 \sum_{j=1}^{\infty} (j+1) \delta_{-j+n,0} e_j^c + 2 \sum_{j=1}^{\infty} (j-1) \delta_{j+n,0} e_{-j}^c \\
 &= -2(n+1) e_n^c + 2\delta_{n,0} e_0^c + d^1(0) e_n^c + \dots + d^1(n-1) e_1^c \\
 &\quad - (e_{n-1}^c d^1(1) + \dots + e_1^c d^1(n-1)) - e_0^c d^1(n) \\
 &= -d^1(n) e_0^c + e_n^c d^1(0) + 2\delta_{n,0} e_0^c.
 \end{aligned}$$

Relation $[c(m), S] = 2(1 - \delta_{m,0}) e_m^c$ follows directly from the definition of the operator S . Next we have

$$\begin{aligned}
 [\overline{W}(n), S] &= \left(\sum_{k \in \mathbb{Z}} [c(k) c(n-k), S] \right) + 2n [c(n-1), S] \\
 &= - \sum_{k \in \mathbb{Z}} (c(k) e_{n-k}^c + c(n-k) e_k^c) - 2n e_{n-1}^c \\
 &= -4(De^c)_n - 4n e_{n-1}^c = 0
 \end{aligned}$$

The proof follows. \square

Now, we will see that in the case $r = 1$ our operator S is a multiple of the screening operator S_1 from [3]:

Corollary 7.2. *Let $r = 1$, and consider $S : W_{p,1} \rightarrow W_{p,-1}$. Then S commutes with the action of the $W(2,2)$ -algebra:*

$$[S, W(n)] = [S, L(n)] = 0 \quad (n \in \mathbb{Z}).$$

Moreover, S is a $W(2,2)$ -homomorphism which is proportional to S_1 .

Proof. In the case $r = 1$ we have that $d^1(0)$ and $Q = e_0^c$ act trivially on $W_{p,1}$, and therefore Lemma 7.1 implies that

$$[S, W(n)] = [S, L(n)] = 0 \quad (n \in \mathbb{Z}).$$

It is clear that the $W(2,2)$ -homomorphism $S_1 : W_{p,1} \rightarrow W_{p,-1}$ from [3, Section 4] is uniquely determined by the properties

$$[S_1, L(n)] = 0, \quad [S_1, I(n)] = -e_n^c,$$

which gives $[S_1, c(n)] = \frac{1}{c_{L,I}} e_n^c$. Now Lemma 7.1 gives that $S = 2c_{L,I} S_1$. \square

7.2. On the Banerjee, Jatkar, Mukhi, Neogi's free field realization of the BMS_3 -algebra. Recently, Banerjee, Jatkar, Mukhi and Neogi in [11] have discovered a new free field realization of the $W(2,2)$ -algebra for central charge $c_L = 26$. The vertex algebra $L^{W(2,2)}(26, c_W)$ is realized inside of the $\beta\gamma$ system. Since the $\beta\gamma$ -system can be embedded into the vertex algebra $\Pi(0)$, one may try to extend this realization in order to obtain an arbitrary central charge c_L . Quite surprisingly, even in the case of the larger vertex algebra $\Pi(0)$, one gets the $W(2,2)$ -structure only for $c_L = 26$.

Recall the definition of following Virasoro vector of central charge $c_L \in \mathbb{C}$.

$$(7.28) \quad \omega = \frac{1}{2}c(-1)d(-1) + \frac{c_L - 2}{24}c(-2) - \frac{1}{2}d(-2).$$

We shall now deform this vector in a different way:

Lemma 7.3. *For every $\mu \in \mathbb{C}$*

$$\tilde{\omega} = \omega + \mu e_{-4}^{-c} \mathbf{1} = \omega + \frac{\mu}{6} D^3 e^{-c}.$$

is a Virasoro vector of central charge c_L .

The proof of lemma follows from a more general statement (which is also noticed in [11]):

Claim: *Assume that $(V, Y, \mathbf{1}, \omega)$ is a VOA of central charge c , and v is a primary, commutative vector of conformal weight -1 . Then $\tilde{\omega} = \omega + \frac{1}{6} D^3 v$ is a Virasoro vector of central charge c .*

The following result is obtained in [11, Section 2]. We include a proof of this result from which one can see that such construction works only for $c_L = 26$.

Proposition 7.4. [11]. *The vertex algebra $L^{W(2,2)}(c_L, c_W)$ for $c_L = 26$ is isomorphic to a vertex subalgebra of $\Pi(0)$ generated by*

$$\begin{aligned} \tilde{\omega} &= \frac{1}{2}c(-1)d(-1) + \frac{c_L - 2}{24}c(-2) - \frac{1}{2}d(-2) + \frac{\mu}{6} D^3 e^{-c} \\ w &= (d(-1) + \frac{c_L - 14}{12}c(-1))e^c \end{aligned}$$

where

$$\mu = -\frac{c_W}{4}.$$

Proof. By direct calculation we get

$$(7.29) \quad w_0 w = \frac{c_L - 26}{3} c(-1) e^{2c} = \frac{c_L - 26}{6} D e^{2c}$$

$$(7.30) \quad w_1 w = \frac{c_L - 26}{3} e^{2c}$$

$$(7.31) \quad w_n w = 0 \quad (n \geq 2).$$

By using formulas

$$(7.32) \quad [L(n), c(m)] = -mc(n+m) + (m^2 - m)\delta_{m+n,0}$$

$$(7.33) \quad [L(n), d(m)] = -md(n+m) - \frac{c_L - 2}{12}(m^2 - m)\delta_{m+n,0}$$

we get that

$$\begin{aligned} L(1)w &= \left([L(1), d(-1)] + \frac{c_L - 14}{12} [L(1), c(-1)] \right) e^c \\ &= \left(2 - \frac{c_L - 2}{6} + \frac{c_L - 14}{6} \right) e^c \\ &= 0, \end{aligned}$$

which easily implies that

$$(7.34) \quad \tilde{L}(n)w = 2\delta_{n,0}w \quad (n \geq 0).$$

Since

$$\tilde{L}(2)w = -\mu e_0^{-c}w = -2\mu = \frac{c_W}{2},$$

we get

$$(7.35) \quad \tilde{L}(n)w = 2\delta_{n,0}w + \frac{c_W}{2}\delta_{n,2} \quad (n \geq 0).$$

Claim now follows from (7.29)-(7.31), (7.35) and Lemma 7.3. \square

Remark 7.5. *It would be interesting to investigate the structure of $W(2, 2)$ -modules $\Pi(p, r)$ with this new action. Since*

$$\widetilde{L(n)} = L(n) - \frac{1}{6}(n+1)n(n-1)\mu e_{n-2}^{-c},$$

we have that $\Pi(p, r)$ are weight $L^{W(2,2)}(c_L, c_W)$ -modules on which $\widetilde{L(0)} = L(0)$. But Whittaker $\Pi(0)$ -module Π_λ is a logarithmic $L^{W(2,2)}(c_L, c_W)$ -module. In our forthcoming papers we plan to investigate the appearance of these modules in the fusion rules analysis at $c_L = 26$.

APPENDIX A. FIGURES

Here we present some visualizations of modules $\Pi(p, r)$ and $\widetilde{\Pi(p, r)}$.

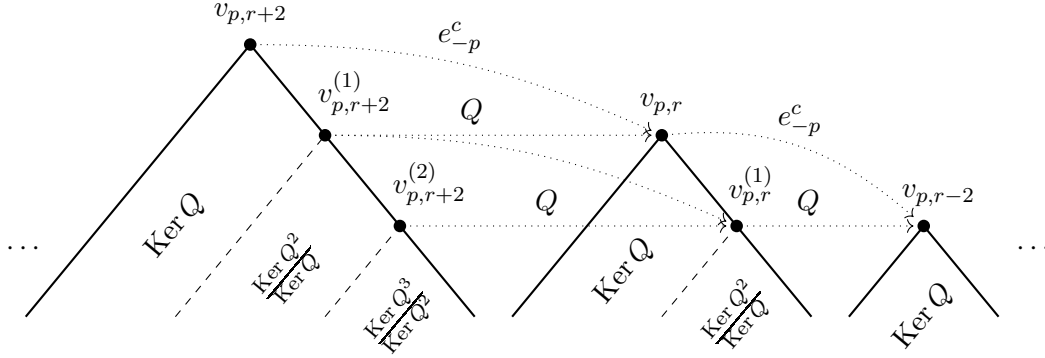


FIGURE 1. $\overline{\Pi(0)}$ -module $\Pi(p, r)$, $p \in \mathbb{Z}_{>0}$

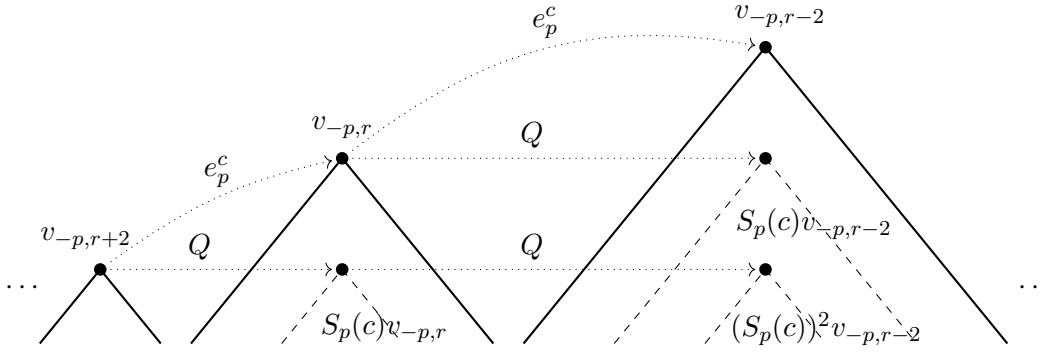


FIGURE 2. $\overline{\Pi(0)}$ -module $\Pi(-p, r)$, $p \in \mathbb{Z}_{>0}$

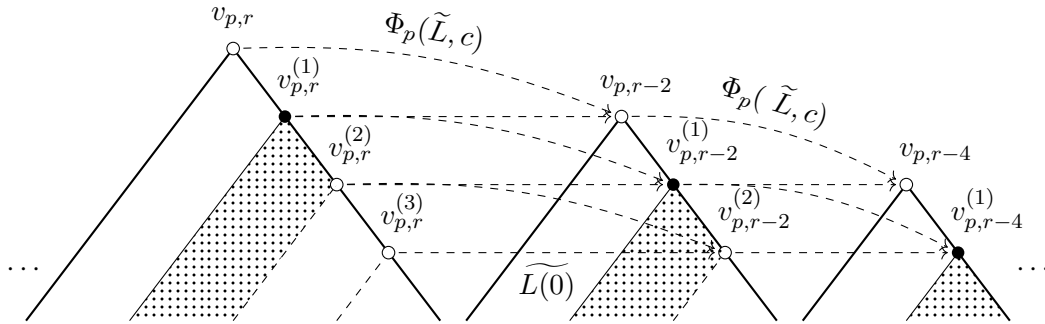


FIGURE 3. Deformed action of \mathcal{H} on $\widetilde{\Pi(p, r)}$, $p > 0$. Dotted area represents a cyclic submodule of $\Pi(p, r)^{(1)}/\Pi(p, r)^{(0)}$ generated by $v_{p,r}^{(1)}$ which is isomorphic to $\widetilde{W_{p,r-2}} \cong V^{\mathcal{H}}(h_{p,r} + p, h_I)$. Arrows represent $\Phi_p(\tilde{L}, c)$ (descending), and $\tilde{L}(0)$ (horizontal).

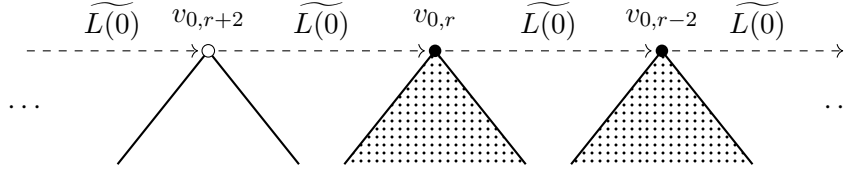


FIGURE 4. $\widetilde{\Pi(0)}$ -module $\widetilde{\Pi(0,r)}$. Dotted area represents a portion of a deformed \mathcal{H} -module $\widetilde{W_{0,r}}$.

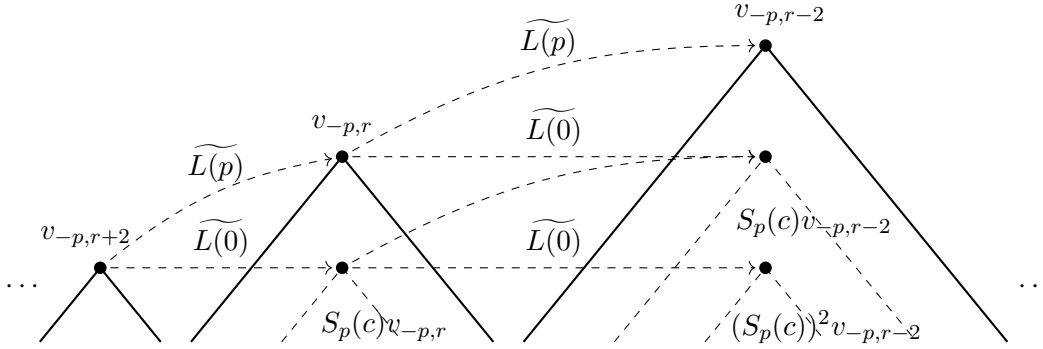


FIGURE 5. Deformed action of \mathcal{H} on $\widetilde{\Pi(-p,r)}$, $p \in \mathbb{Z}_{>0}$

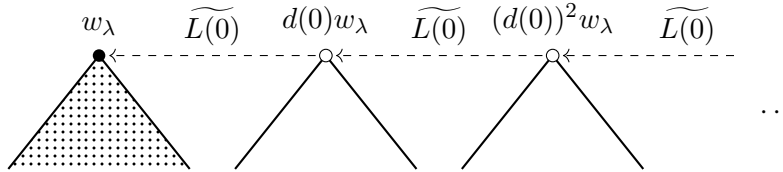


FIGURE 6. A deformed Whittaker module $\widetilde{\Pi}_{\lambda}$. Dotted area represents $\widetilde{\Pi}_{\lambda}^{(0)}$ which is a self-dual highest weight \mathcal{H} -module.

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